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A variational theory for frictional flow of fluids in inhomogeneous porous systems

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Abstract

For nonlinear steady paths of a fluid in an inhomogeneous isotropic porous medium a Fermat-like principle of minimum time is formulated which shows that the fluid streamlines are curved by a location dependent hydraulic conductivity. The principle describes an optimal nature of nonlinear paths in steady Darcy's flows of fluids. An expression for the total resistance of the path leads to a basic analytical formula for an optimal shape of a steady trajectory. In the physical space an optimal curved path ensures the maximum flux or shortest transition time of the fluid through the porous medium. A sort of "law of bending" holds for the frictional fluid flux in Lagrange coordinates. This law shows that – by minimizing the total resistance – a ray spanned between two given points takes the shape assuring that its relatively large part resides in the region of lower flow resistance (a 'rarer' region of the medium). Analogies and dissimilarities with other systems (e.g. optical or thermal ones) are also discussed.

Keywords: Variational principles; Frictional flows; Inhomogeneous media; Darcy's law

1. Introduction

Darcy's Law states that the rate of fluid flow through a porous medium is proportional to the gradient of hydraulic head within that fluid. This law is the basic constitutive relationship that describes fluid motion in porous media and that helps us to understand the movement of fluids in the Earth's crust (Fig. 1). The constant of proportionality between the flow and the gradient of hydraulic head is the hydraulic conductivity; a property of both the porous medium and the fluid moving through this medium. When this conductivity is a known nonlinear function of a length coordinate x (set here purposely in the direction parallel to the conductivity gradient) a Fermat-like principle can be formulated for steady frictional flows in isotropic media, simple and composite. In contemporary modeling of fluid's seepage acceleration terms and corresponding transients can be included in generalized form of Darcy's law, yet they are neglected in the presented variational description for the fluid transport in an inhomogeneous porous medium. Still, there appears the need to take into account the effect of the state changes in the coefficients of the (steady) variational formulation. In Darcy's law, the state-dependent hydraulic conductivity is such a parameter sensitive for the state changes. In this work, we consider a steady, frictional flow of the fluid in an inhomogeneous porous medium. We assume that the hydraulic conductivity is a known prescribed function of the spatial co-ordinate x. While in contemporary modeling of fluid's seepage acceleration terms and corresponding transients can be included in generalized form of Darcy's law, they are neglected in the presented variational description for the fluid transport. The basic purpose of the present research follows from the observation that, for various reasons, packing can be inhomogeneous (causing location dependent hydraulic resistances). Transport of the fluid through the porous medium is then associated with curved fluid's streamlines, hence the rationale for a variational principle that could treat the curvature effects quantitatively. According to the principle, the shape of a steady trajectory

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Nomenclature

A	variable area perpendicular to fluid flow
A_0	constant transfer area projected on axis y
c	bending constant for a frictional ray
g	gravity acceleration
H	Hamiltonian function
h	hydraulic head, $P(\rho_m g)^{-1} + z$
I	total fluid flux passing through the system
k	conductivity related to gradient of hydraulic head
l	length coordinate
p	momentum type integral or $\bullet R/\bullet y$
P	pressure
$R^{1,2}$	total resistance of frictional path between points 1 and 2
$R(x,y)$	minimum resistance potential
t	physical time
u	controlled direction dy/dx
x	coordinate perpendicular to the resistance gradient
y	coordinate tangent to the resistance gradient
z	vertical coordinate
α	angle
ρ	specific frictional resistance or reciprocal of conductivity k
ρ_m	mass density of the fluid

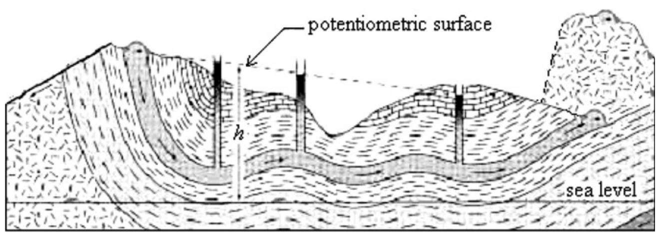

Fig. 1. Hydraulic head: water level in the wells measures the head level in the aquifer.

of nonlinear frictional flow is the result of the maximum flux or shortest transition time of the fluid through the medium. This property leads to the prediction of shapes of related "rays" or paths of the fluid flow. This also leads to the process description in terms of wave-fronts and related Hamilton–Jacobi theory which is derived in this paper from the optimization algorithm of the dynamic programming method. In a part our approach transfers to the realm of nonlinear Darcy's flows some results obtained in earlier treatments of fluid transfer in the energy and entropy representation [1,2]. However, instead of performing the analysis in the realm of paths, we concentrate on the wave-front description of the fluid flow and the corresponding Hamilton–Jacobi theory. The relevant physical picture refers to tracing of fluid propagation in Lagrange coordinates, where the fluid's flow through the porous skeleton is attributed to motion of the same fluid particles rather than to the fluid's passage through a fixed region of the physical space. Applications of functional equations and the Hamilton-Bellman–Jacobi equation are effective when Bellman's method of dynamic programming [3] is applied to wave-fronts. That method involves an alternative view: the observation of mass conduction in porous medium in terms of wave-fronts rather than fluid paths. In this method potential functions describing the minimum resistance or the minimum transition time are obtained in an explicit way by analytical or numerical approaches.

Following an earlier work [1], but working in the entropy representation as the only correct one that allows the inclusion of conserved energy transfer [2], one derives a sort of law of refraction for fluid's frictional transfer. The derivation originates from the boundary conditions that express the continuity of the transfer potential (hydraulic head) and the mass flux continuity through the interface. In this approach we assume a steady state frictional flow of a fluid through a smooth interface separating two regions of different hydraulic conductivity without interfacial resistance and with no sources or sinks of the fluid. (The assumption of the interface transport is only for technical reasons as explained at the end of this section.) Under these conditions hydraulic head $h = P(\rho_m g)^{-1} + z$ is continuous across the interface and the path of the frictional flux ("the Darcy's ray") at each point is uniquely determined by the direction of the gradient of the hydraulic head $h = P(\rho_m g)^{-1} + z$. Then one shows that a passage of a conserved entity, such as the fluid's mass flux, through an interface is governed by a so-called tangent law of refraction. Whereas flows of other quantities, associated with non-conserved fluxes, are governed by laws deviating from the tangent law. In particular, flows of photons are described by sine law of refraction. Thus, we analyse here the nature of conserved flows in both discrete and continuous cases (Section 2 and 3, respectively). We also derive, in Section 4, the variational setting and Hamilton–Jacobi theory for wave-fronts. Section 5 describes a relation between the present formalism and those other ones describing different physical systems (e.g. those with non-conserved flows), and Section 6 provides an example for a porous system transport. Section 7 specifies conclusions regarding properties of optimal paths in inhomogeneous porous systems and shows what effect our results could have on the practical formulation of problems on flows of fluids in inhomogeneous media based on Darcy's law. The results imply applications of the theory to various practical systems with mass transfer (e.g. to porous drying

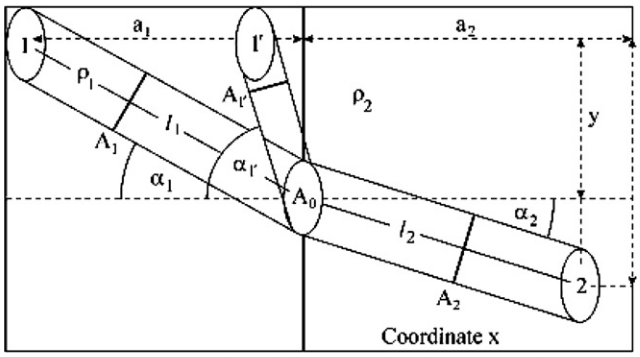

Fig. 2. Fermat-like principle for nonlinear Darcy's flows. Two cases of the incident ray bending correspond with the fluid motion for two different resistance ratios ρ_1/ρ_2 at a constant gradient of h and specific resistance ρ_2.

systems) in which a non-uniform distribution of fluid's mass flux is observed, caused by local variations of porosity and hydraulic resistance ("imperfect systems").

It should be understood that the discrete model of mass transport through an interface between two homogeneous media, is, in fact, only a discrete tool enabling a suitable treatment of an inhomogeneous porous system. The physically significant entity is the continuous inhomogeneous limit of the discrete model, that corresponds to an infinite number of infinitesimal discrete steps, each step involving two neighbouring, infinitesimally narrow, vertical homogeneous layers of a porous system, such as in Fig. 2. Both neighbouring layers differ slightly in their physical properties (e.g. in their specific hydraulic conductance), so that a real porous system is treated as an arrangement composed of an infinite number of infinitesimally narrow vertical layers. The results refer to flows following curved paths caused by weak non-linear terms, these terms representing only the effect of state variations (in coefficients) without variations in state derivatives ("quasilinear models"). The trajectory bending is just the effect of change of the porous system specific conductance with fluid's residence time.

2. A discrete analogue of Fermat problem leading to a bending law

Fig. 1 interprets hydraulic head. Whereas Fig. 2 depicts a frictional flow of the fluid through a vertical interface separating two regions of different hydraulic conductivity. α_1 and α_2 are respectively the angles of incidence and refraction, whereas ρ_1 and ρ_2 are corresponding resistances of the two porous media. The reciprocals of resistances, k_1 and k_2, are hydraulic conductances related to gradients of the hydraulic heads. For an interface characterized by good contact between media or the continuity of the quantity h across the interface, the boundary conditions can be written in the following form (IaI refers to the absolute value of a)

$$|\nabla h|_1 \sin\alpha_1 = |\nabla h|_2 \sin\alpha_2 \tag{1}$$

for the tangent component of and

$$k_1|\nabla h|_1 \cos\alpha_1 = k_2|\nabla h|_2 \cos\alpha_2 \tag{2}$$

for the normal component in absence of sources and sinks of mass. The law of the refraction for the conduction rays follows as ratio of these equations

$$\rho_1 \tan\alpha_1 = \rho_2 \tan\alpha_2 \tag{3}$$

where $\rho = 1/k$ is the specific hydraulic resistance of the porous system. This equation is analogous to Snell's law with the tangent replacing the sine and the flow resistivity replacing the refractive index. This law was originally formulated for the heat flow and reciprocal of the usual thermal conductivity λ [1] which we however attributed in [2] to the reciprocal of Onsagerian conductivity k as the more proper quantity than λ because of the basic link of k with the entropy production.

Examining the deviation of the refracted ray from the incident ray it is convenient to introduce the relative hydraulic resistance of the second porous layer with respect to the first; $\beta = \rho_2/\rho_1$. In terms of β and the angle of incidence, α_1, the angle of refraction is

$$\alpha_2 = \arctan(\tan\alpha_1/\beta) \tag{4}$$

whereas the angle of deviation of the reflected ray, $\Delta = \alpha_2 - \alpha_1$ equals

$$\Delta \equiv \alpha_2 - \alpha_1 = \arctan(\tan\alpha_1/\beta) - \alpha_1. \tag{5}$$

Figures were obtained [1] that depict the angle of refraction, α_2, and deviation, Δ, as functions of the incidence angle α_1 and the resistance ratio, $\beta = \rho_2/\rho_1$. The case $\beta > 1$ is the optical analogue of refraction from the rarer to the denser medium, and inversely. In contrast to Snell's refraction law in optics, where the deviation Δ decreases monotonically with the incidence angle α_1, in the tangent law case the deviation attains an extremum and approaches zero for both normal and grazing incidences, $\alpha_1 = 0$ and $\alpha_1 = 90°$, respectively. In the limit as β tends to zero, the incidence angle α_1 approaches 0° which means that the frictional flux entering a perfect hydraulic conductor is perpendicular to the interface (a geometric surface separating two regions with different hydraulic conductivities). Otherwise, as β tends to infinity, the refraction angle α_2 becomes zero, which means that in the limit of perfect insulator infinitesimally small Darcy's flux emerges perpendicularly to the "interface".

In the flow tube in Fig. 2 the fluid is transferred along the length dl by the cross-section perpendicular to the mass flux. The perpendicular cross-section has the area A which may change with l; the volume differential $dV = A\,dl$. As distinguished from more standard treatments, we integrate here over the volume V 'moving with the fluid'; in this case x and y are Lagrange coordinates and the fluid's flow is attributed to motion of the same portion of the fluid rather than to flow through a fixed area in the space. We introduce the fluid current $I = dQ/dt$ as the amount (mass or volume) of the fluid transferred by the system per unit time. The current I is the conserved property, i.e. $I_1 = I_2 = I$. The related density of fluid flux is $dQ/A\,dt$ or $J_q = I/A$. We also use here the resistance ρ which is the reciprocal of the hydraulic conductivity. The corresponding total resistances: $R_1 = \rho_1 l_1/A_1$ and $R_2 = \rho_2 l_1/A_2$ play also a role as we shall see soon. Here A_1 and A_2 are the cross-sectional areas of a "tube" of the Darcy's flux in porous media 1 and 2. The products $R_1 I$ and $R_2 I$ of resistances and current I have units of the hydraulic head, consistent with the fact that they are determined by the difference of h.

Fig. 2 shows decrease of the area A_1 perpendicular to the Darcy's ray with the increased difference between conductivities k_1 and k_2, at a constant k_2. We shall now show the essential role of the area A_1 on the tangent law of bending for the mass flux in a porous medium whose hydraulic resistance exhibits a jump through the interface. The Darcy's ray travels between two fixed points, 1 and 2. If A_0 is the constant area of a flux tube intercepted by the interface (the constant area of projection of the mass flux tube cross-sectional area on the surface of constant resistance), then the cross-sectional areas of the flux tubes in two porous media are

$$A_1 = A_0 \cos\alpha_1, \qquad A_2 = A_0 \cos\alpha_2. \tag{6}$$

Thus the total resistance for the fluid's flow between the points 1 and 2 is

$$R^{1,2} = \frac{\rho_1 l_1}{A_1} + \frac{\rho_2 l_2}{A_2} = \frac{1}{A_0}\left(\frac{\rho_1 l_1}{\cos\alpha_1} + \frac{\rho_2 l_2}{\cos\alpha_2}\right). \tag{7}$$

Substituting to this equation the values of $\cos\alpha_1$ and $\cos\alpha_2$ from Fig. 2 as

$$\cos\alpha_1 = \frac{a_1}{\sqrt{a_1^2 + y^2}}, \qquad \cos\alpha_2 = \frac{a_2}{\sqrt{a_2^2 + (L-y)^2}} \tag{8}$$

and the lengths l_1 and l_2 as

$$l_1 = \sqrt{a_1^2 + y^2}, \qquad l_2 = \sqrt{a_2^2 + (L-y)^2} \tag{9}$$

one finds

$$R^{1,2} = \frac{1}{A_0}\left(\frac{\rho_1 (a_1^2 + y^2)}{a_1} + \frac{\rho_2\left(a_2^2 + (L-y)^2\right)}{a_2}\right). \tag{10}$$

We stress that it is the vertical coordinate y of the intersection point with the interface which is allowed to vary; the horizontal coordinate x of that point is always constant and equal a_1. Since $y/a_1 = \tan\alpha_1$ and $(L-y)/a_2 = \tan\alpha_2$, the condition requiring the first derivative $dR_{1,2}/dy$ to vanish

$$\frac{dR^{1,2}}{dy} = \frac{2}{A_0}\left(\frac{\rho_1 y}{a_1} - \frac{\rho_2 (L-y)}{a_2}\right) = 0 \tag{11}$$

is equivalent with the requirement that the tangent law, Eq. (3), is satisfied. Differentiating Eq. (11) with respect to y once again, one obtains

$$\frac{d^2 R^{1,2}}{dy^2} = \frac{2}{A_0}\left(\frac{\rho_1}{a_1} + \frac{\rho_2}{a_2}\right) > 0 \tag{12}$$

which proves the minimum property of $R^{1,2}$ at the stationary point. Consequently the postulate that the fluid follows the trajectory of least resistance is the correct physical principle that leads to the tangent law (3), implied by boundary conditions (1) and (2). Eqs. (1) and (2), may be seen as the consequence of the mass conservation and the principle of least resistance. (As the latter is incorporated in the theorem of the least entropy production, one may regard Eqs. (1) and (2) as the consequence of the minimum entropy production applied to any conserved mass flux in a purely frictional flow.) In what follows we shall describe an extension of the above (discrete) problem to the continuous situation where an inhomogeneous porous material with a variable local resistance $\rho(x)$ is regarded as a sequence of many vertical infinitesimal layers, and the frictional flow of the fluid occurs through a porous material whose hydraulic conductivity varies continuously with x.

3. Variational continuous paths for fluid flow in inhomogeneous porous systems

When describing a fluid flow within an inhomogeneous porous structure we shall use a corresponding continuous description that uses a special reference frame (x, y). In this frame the local resistance of the fluid's frictional flow changes along the axis x (Fig. 2), and the axis y is tangent to a surface of constant specific resistance ρ. The derivative $u = dy/dx$, formally a control variable, coincides with the local direction of the gradient of hydraulic head h. As in the discrete analysis above, in the continuous system we assume the layered structure of the material with many thin layers arranged vertically. To derive optimality conditions for the nonlinear Darcy's flows in the multilayer system composed of N layers, the optimization theory (Bellman's optimality principle, [3–6]) admits that one can consider two arbitrary layers under assumption of their fixed states at the left and right boundaries, whereas the state between the considered layers must be free, to allow an optimization. Keeping in mind the analogy between thermal and frictional problems, the shape of Darcy's rays can be described as an optimal control problem for a minimum of the total resistance [2]. Certain specific coordinates x and y are usually applied to describe the problem locally; the axis y is parallel to the surfaces of constant resistance. In this case a general structure of the variational principle is represented by an equation

$$\delta J = \delta \int_{l_1}^{l_2} p\, dl = \delta \int_{x_1}^{x_2} p(x, y, u)\left(\sqrt{1+u^2}\right) dx = 0 \tag{13}$$

where $u = dy/dx$ is the slope of the tangent to the path. In this reference frame the local resistance of the fluid flow changes along the axis x, the axis y is tangent to a surface of constant resistivity $\rho = C$ and $u = dy/dx$ is the local direction of the gradient of hydraulic head. An analysis shows that the function p of the Darcy's rays corresponds with the function

$$p(x,y,u) = A_0^{-1}\rho(x)\sqrt{1+u^2}. \tag{14}$$

Eq. (13) with function $p(x,y,u)$ defined by Eq. (14) should then be optimized with respect to the control $u = dy/dx$ within each infinitesimal layer. In Eq. (14) A_0 is the constant area of projection of the heat flux tube cross-sectional area on the surface of constant resistance. The reader is referred to the literature [2] for details regarding numerical studies of Eq. (13) by the dynamic programming method. An important conclusion should be kept in mind when stressing differences between propagation of Darcy's and optical rays: while the simplest optical rays are described by Euclidean and Riemmanian geometry, it is rather Finslerian geometry [7] that is valid for Darcy's rays. The essence of this geometry is an explicit dependence of the metric tensor on directions, a property that enables one to include non-isotropic systems into considerations.

To derive the tangent law of bending for continuous Darcy's rays let us regard the porous system as a pseudo-continuum and assume a variable hydraulic resistance ρ. In our coordinate system the planes of constant resistance ρ are perpendicular to the axis x, i.e. the flow resistance is a continuous function of x only. Let us imagine that we rotate the system about the axis x until the gradient of the hydraulic head is parallel to the x–y plane. A set of flux tubes with the flowing fluid matter can be defined as in the discrete problem analysed above. While in the discrete problem the total resistances are $R_1 = \rho_1 l_1/A_1$ and $R_2 = \rho_2 l_2/A_2$, in the continuous problem these relations are represented jointly by the single local relationship, $dR = \rho A^{-1} dl$, in which A is the variable cross-sectional area of a "tube" of mass flux perpendicular to the flow in a inhomogeneous medium. The products $R_1 I$ and $R_2 I$ have the units of length and describe the differences of the hydraulic head.

As in the discrete problem, we postulate that the path of the fluid in porous medium flowing between two fixed points 1 and 2 is that along which the total resistance is a minimum. Similarly to the discrete problem, $A = A_0\cos\alpha$, where A_0 is the constant (x-independent) area of the flux tubes intercepted by the interface. The variable cross-sectional area of the flux tube in the medium is described by a continuous counterpart of Eq. (6)

$$A = A_0\cos\alpha \tag{15}$$

(see Fig. 2) whereas the incidence angle varies with x according to the formula

$$\cos\alpha = \frac{dx}{dl} = \frac{dx}{\sqrt{dx^2+dy^2}} \tag{16}$$

In the above equations α is the angle between the gradient of the hydraulic head (or the Darcy's ray) Eqs. (4), (13) and (15) then imply the formulation

$$\begin{aligned}\delta\int_{l_1}^{l_2}\frac{\rho}{A}\,dl &= \delta\int_{l_1}^{l_2}\frac{\rho(x)}{A_0\cos\alpha}\,dl\\ &= \delta\int_{l_1}^{l_2}\frac{\rho(x)\sqrt{dx^2+dy^2}}{A_0\,dx\left(\sqrt{dx^2+dy^2}\right)^{-1}}\\ &= \delta\int_{l_1}^{l_2}A_0^{-1}\rho(x)(1+(dy/dx)^2)\,dx = 0\end{aligned} \tag{17}$$

which describes the vanishing first variation for the functional of total resistance defined as

$$R^{1,2} \equiv \int_{l_1}^{l_2}A_0^{-1}\rho(x)(1+(dy/dx)^2)\,dx \tag{18}$$

Comparison of Eqs. (13) and (17) proves that in Darcy's flows the function $p(x,y,u)$ has the form consistent with Eq. (14). The dependence of p on u is caused by the change of the area A perpendicular to the flow with x, as shown in Fig. 2.

When a pressure field in a isotropic porous medium is imposed along with fixing the hydraulic head gradient, the flow of the fluid in a permeable porous skeleton can be described in terms of 'rays', or paths of fluid flow in direction of hydraulic head gradient. Their deviation from straight lines results from a variable conductivity. In fact, frictional rays travel along paths satisfying the principle of minimum of entropy production [2], which seems at first glance quite different than the well-known Fermat principle of minimum time for optical rays. However, the principle assures the minimum resistance of the path, which, in a dual problem, causes the maximum of fluid flux through the porous medium and makes the residence time of the fluid in this medium as short as possible. This is quite similar to Fermat principle for propagation of light [8]. Our purpose is to investigate this phenomenon by methods of optimization. We use a particular reference frame (x,y) in which the local resistivity of fluid flow changes along the axis x (Fig. 2), the axis y is tangent to a surface of constant specific resistivity ρ and $u = dy/dx$ is the local direction of the gradient of the hydraulic head. A family of frictional paths entering at various angles α_1 is considered corresponding with various gradients of the hydraulic head. To determine each path as an extremal is the goal of the variational approach applied. (The approach can also be extended to handle the energy flow in layered composites [2].) The shape of Darcy's ray caused by non-uniform resistance $\rho(x)$ can be described as an optimal control problem for a minimum of total resistance

$$J = \int_{x_1}^{x_2}A_0^{-1}\rho(x)(1+u^2)\,dx \tag{19}$$

In the frictional Darcy's flow the resistance ρ is the reciprocal of the conductivity related to the gradient of the hydraulic head (grad$T$ in the case of energy flow, [2]). A is a variable area perpendicular to fluid flow and A_0 is a constant transfer area projected on axis y. A path or flow or "ray" bends depending on the ratio of ρ_2/ρ_1; the bending constant for a ray equals c. Eq. (22) below defines c in terms of a function R describing the minimum total resistance.

The total resistance of an *optimal* path linking cross-sections 1 and 2 is described by the minimum resistance function $R(x, y)$. Eq. (1) should be optimized with respect to the slope $u = \mathrm{d}y/\mathrm{d}x$ as a control variable within each differential layer $\mathrm{d}x$. For a discussion of the role of functional (19) in energy transfer problems the reader is referred to literature [1,2]. Essential for the functional simplicity is the suitable frame (x, y) in which the local resistance of the fluid flow changes along the axis x whereas diverse frictional paths may enter the investigated region at various angles α_1, (and various controls $u = \tan\alpha = \mathrm{d}y/\mathrm{d}x$), where each angle corresponds to a different gradient of the hydraulic head and hence to a different flux density of the flowing fluid.

4. Hamilton–Jacobi–Bellman theory

For a porous system described as a pseudo-continuum whose specific resistance changes along axis x, a minimum resistance function of the problem is defined as

$$R(x_1, y_1, x_2, y_2) \equiv \min \int_{x_1}^{x_2} A_0^{-1}\rho(x)(1+u^2)\,\mathrm{d}x, \tag{20}$$

where the integrand describes the local resistance. The state or potential function R satisfies an important equation of the optimization theory called the Hamilton–Jacobi–Bellman equation (HJB equation, [4–6]). A general HJB equation is a quasilinear partial differential equation with the extremization sign with respect to the optimal control u. The function R appears in the HJB equations explicitly, whereas it is only implicit in the ordinary canonical equations of the Pontryagin's maximum principle [5]. In the latter case, R is hidden in the so-called adjoint variables (components of the gradient vector of R). In thermodynamics, economics or some other disciplines approaches involving HJB equations may be superior with respect to the Pontryagin's method whenever the principal function R has to be determined.

Usually HJB equations are derived in an exact and systematic way by the dynamic programming method [4–6]. However, in this paper a simple and brief while insightful method based on Caratheodory's lines of reasoning is outlined [6]. From the definition of the minimum resistance function R in functional (20) we obtain

$$\max_{u(t)}\left\{R(x_1, y_1, x_2, y_2) - \int_{x_1}^{x_2} A_0^{-1}\rho(x)(1+u^2)\,\mathrm{d}x\right\} = 0. \tag{20a}$$

After introducing the total derivative of optimal characteristic function R with respect to a variable upper limit of integration

$$\max_{u(t)}\left\{\int_{x_1}^{x_2}\left[\frac{\mathrm{d}R(x_2, y_2)}{\mathrm{d}x} - A_0^{-1}\rho(x)(1+u^2)\right]\mathrm{d}x\right\} = 0. \tag{20b}$$

This describes a vanishing maximum of the negative resistance (negative integrand of Eq. (20)) gauged by the total derivative of optimal function, $\mathrm{d}R/\mathrm{d}x$, at the state 2. Expanding the total time derivative of R in terms of the partial derivatives, $\partial R/\partial x$ and $\partial R/\partial t$, the derivative of R with respect to time can be taken off this equation and index 2 can be omitted for variable final states. This procedure shows that the optimal function R with satisfies a quasilinear partial differential equation

$$\frac{\partial R}{\partial x} + \max_u\left\{\frac{\partial R}{\partial y}u - A_0^{-1}\rho(x)(1+u^2)\right\} = 0. \tag{21}$$

Eq. (21) is called the Hamilton–Jacobi–Bellman equation (HJB equation) of the problem. The formalism represented by Eqs. (20) and (21) can also be derived from the continuous algorithm of the dynamic programming method applied to the performance criterion (19), the derivations are then slightly more complicated and longer [4].

For Eq. (21) the resistance ρ is independent of y, and the HBJ equation is linear rather than quasilinear. Extremizing the Hamiltonian expression in Eq. (21) yields an optimality condition for the fluid flow within each infinitesimal layer. This optimality condition is written below in the form of the tangent law of bending for a Darcy's ray

$$\rho(x)\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{A_0}{2}\frac{\partial R}{\partial y} \equiv c, \tag{22}$$

where c is a constant which may be both positive or negative. The constancy of the partial derivative $\partial R/\partial y$ follows from an explicit independence of density ρ in Eq. (19) with respect to y. A suitable integral formula for bending constant in terms of deviation $y - y^0$ follows in the form

$$c = (y - y^0)\left(\int_{x^0}^{x}\rho^{-1}(x')\,\mathrm{d}x'\right)^{-1}. \tag{23}$$

The optimal control u is the solution (22) of the HJB equation (21) in terms of $p = \partial R/\mathrm{d}y$. Substituting this solution into Eq. (21) yields the Hamilton–Jacobi equation for the frictional fluid flow in an arbitrary layer

$$\frac{\partial R}{\partial x} + A_0^{-1}\rho(x)\left(\left(\frac{A_0}{2\rho(x)}\frac{\partial R}{\partial y}\right)^2 - 1\right) = 0, \tag{24}$$

where the second term of the left hand side expression is the *optimal* Hamiltonian. The solution to the above equation can always be broken down to quadratures, see Eq. (26) below. However, if the function of specific resistivity $\rho(x)$ is too complicated, the integrals cannot be evaluated analytically. This difficulty calls for a discrete approach that solves numerically the associated Bellman's recurrence equation of the problem. Under the thin layer assumption this equation has the form

$$R^n(y^n, x^n) = \min_{u^n,\theta^n} \Big\{ A_0^{-1}\rho(x^n)(1+(u^n)^2)\theta^n + R^{n-1}(y^n - u^n\theta^n, x^n - \theta^n) \Big\}, \tag{25}$$

where $\theta^n = x^n - x^{n-1}$. This still cannot be analytically solved for an arbitrary $\rho(x)$, thus a dynamic programming sequence of R^n must be generated numerically. Yet, in the limit of an infinite number of stages, an analysis shows that the potential function satisfying Eq. (21) takes the limiting form

$$R(x,y) = \int_{x_0}^{x} A_0^{-1}\rho(x')\,\mathrm{d}x' + A_0^{-1}(y-y^0)^2\left(\int_{x_0}^{x}\rho^{-1}(x')\,\mathrm{d}x'\right)^{-1}. \tag{26}$$

This function satisfies both HJB equation (21) and Hamilton–Jacobi equation (24). In fact, the analytical solution of the continuous problem is given quite generally by Eq. (26), that is valid for an arbitrary function $\rho(x)$ and that includes, of course, the case of the linear ρ as well. In an example of Section 6 the case of exponential ρ is discussed in some detail.

5. Comparison with other physical systems

We shall now consider extensions including non-dissipative flows in various systems, e.g. optical and mechanical ones. When A is the area of the cross-section perpendicular to the flow and coordinates x and y are applied the general form of the variational principle is given by Eq. (27)

$$\delta\int_{l_1}^{l_2} p\,\mathrm{d}l = \delta\int_{x_1}^{x_2} p(x,y,u)\sqrt{1+u^2}\,\mathrm{d}x = 0, \tag{27}$$

where $u = \mathrm{d}y/\mathrm{d}x$ is the slope of the tangent to the path. In this reference frame the local resistance of the flow ρ changes along the axis x, the axis y is tangent to a surface of constant specific resistance $\rho = X$ and $u = \mathrm{d}y/\mathrm{d}x$ is the local direction of the gradient of certain potential, say Π, which is not necessarily hydraulic head h but it may represent chemical potential of certain species or even linear combination of chemical potentials in the form known for substrate and product terms of chemical affinity [9]. A set of flux tubes with the flowing matter can be defined again. In a discrete problem the total quantities p_k in two neighbour subsystems are $p_1 = I_1^2\rho_1/A_1$ and $p_2 = I_2^2\rho_2/A_2$ where both symbols I_k designate the corresponding currents. In the continuous case these relations are represented jointly by the single local relationship, $p(x) = I^2\rho A^{-1}$, in which A is the variable cross-sectional areas of a "tube" perpendicular to the flux passing an inhomogeneous medium. It may be shown that the products $p_1 l_1$ and $p_2 l_2$ are related to the entropy production in both subsystems. It appears that for non-conserved flows the principle of minimum resistance ceases to hold as an exact law, yet the principle of minimum of entropy production is still valid.

We test the postulate that the path of a generalized (say, chemical) current I flowing between two fixed points 1 and 2 is that along which the total entropy production is a minimum. In the general case the chemical Lagrangian corresponding with the general chemical function $p(x,y,u)$ of Eq. (27) has the form

$$L(x,y,u) \equiv p(x,y,u)\sqrt{1+u^2} = I^2(x,y,u)A^{-1}(u)\rho(x)\sqrt{1+u^2} \tag{28}$$

where $p(x,y,u)$ is the formal analogue of the refraction coefficient of the optical Fermat problem. The optical case and its Hamilton–Jacobi theory are well known [10]. Yet, the case of constant I is that of a conservative flux with $A^{-1}(u) = A_0^{-1}\sqrt{1+u^2}$. In this case $p(x,y,u) \equiv I^2A_0^{-1}\rho(x)\sqrt{1+u^2}$ and the optimization problem is that of the minimum resistance. The dependence of p on u is caused by the change of the area A perpendicular to the flow with x, Fig. 2. It is Finslerian rather than Riemmanian geometry which describes problems of this sort. In fact, the Finslerian geometry refers to "dissipative" rather than to "reversible" Lagrangians L [7].

The Euler–Lagrange equation of the general problem described by Eq. (28) is

$$\sqrt{1+u^2}\frac{\partial p}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{pu}{\sqrt{1+u^2}} + \frac{\partial p}{\partial u}\sqrt{1+u^2}\right\} = 0 \tag{29}$$

This equation admits a simple and important interpretation. If the tangent to the extremal makes the angle α with the axis x, then

$$\frac{\mathrm{d}y}{\mathrm{d}l} = \frac{\mathrm{d}y}{\sqrt{\mathrm{d}x^2+\mathrm{d}y^2}} = \frac{u}{\sqrt{1+u^2}} = \sin\alpha. \tag{30}$$

In the case of a conserved flux (constant I), we apply $p \equiv I^2A_0^{-1}\rho(x)\sqrt{1+u^2}$ and then

$$\sqrt{1+u^2}\frac{\partial p}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{pu}{\sqrt{1+u^2}} + I^2A_0^{-1}\rho(x)u\right\} = 0 \tag{31}$$

or, since $u\left(\sqrt{1+u^2}\right)^{-1}$ is the sine of α

$$\sqrt{1+u^2}\frac{\partial p}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\{p\sin\alpha + I^2A_0^{-1}\rho(x)\tan\alpha\} = \sqrt{1+u^2}\frac{\partial p}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\{2I^2A_0^{-1}\rho(x)\tan\alpha\} = 0. \tag{32}$$

In view of the y-independent L and p the flux I is conserved and Eq. (27) simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}x}(2\rho(x)(\mathrm{d}y/\mathrm{d}x)) = 0 \tag{33}$$

or, since $\mathrm{d}y/\mathrm{d}x = \tan\alpha$

$$\rho(x)\tan\alpha(x) = c \tag{34}$$

where c is a constant. Here we have recovered the tangent law of bending for an inhomogeneous porous medium in which the resistance is a function of x. Eqs. (22) and (34) transfer the discrete result (3) to continuous systems.

On the other hand, in the simple optical case when $p = n$ (the refraction coefficient), p is independent of u and only $L = p\sqrt{1+u^2}$ depends on u

$$\sqrt{1+u^2}\frac{\partial p}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\rho u}{\sqrt{1+u^2}}\right) = 0. \tag{35}$$

Eq. (35) is then reduced to

$$\frac{\partial p}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}l}(p\sin\alpha) = 0. \tag{36}$$

This equation describes a geometrical property of the extremal path, which is independent of the choice of axes. Therefore its interpretation can use any convenient system of axes. As p and n depend on x and y only, the equation $n =$ constant is that of the plane curve and by varying the constant we obtain a family of the curves, the so-called level curves. If O is a point on the extremal, let us take it as the origin and the normal and tangent to the level curve through O as the x and y axes respectively. Now at O the tangent to the curve $n =$ constant is perpendicular to the axis x, hence $(\partial n/\partial x)/(\partial n/\partial y)$ must be infinite and so $\partial n/\partial y = \partial p/\partial y = 0$. In this particular frame Eq. (36) becomes the Snell (sine) law of refraction for optical systems

$$n\sin\alpha = \text{constant}. \tag{37}$$

This frame-dependent result holds for all points along the extremal. Eq. (37) is not restricted to optical case. It is valid also in the mechanical case where the level curves are given by $(h - V)^{1/2} =$ constant, and, as this is equivalent to $V =$ constant, the level curves are curves of the constant potential energy. We observe that frictional and chemical cases are in general more complicated than optical. Yet, while we can formulate diverse descriptions that formally include the well-known optical principle, solid physical information is necessary to attribute a definite formulation to a concrete physical system.

6. An example with exponential resistance

Consider now an example of flow of fluid through a porous system whose hydraulic resistance increases exponentially with the co-ordinate x. We begin with the Euler–Lagrange equation for the functional (19)

$$\frac{\mathrm{d}}{\mathrm{d}x}(2\rho(x)(\mathrm{d}y/\mathrm{d}x)) = 0 \tag{38}$$

that coincides, in this problem, with a differential form of bending formula, Eq. (22). Since the equation for an extremal is the second order differential equation, its solution depends on two integration constants. When the function $\rho(x)$ is known, variables in Eq. (38) can be separated. Integration between a fixed initial point (x^0, y^0) and an arbitrary final point (x, y) yields a general integral

$$y = y^0 + c\int_{x^0}^{x}\frac{\mathrm{d}x'}{\rho(x')}, \tag{39}$$

where the integration constants are c and y^0. This equation is a counterpart of formula (23) and applies when one wants to evaluate trajectory $y(x)$ for a given c rather than inversely. For the exponential increase of hydraulic resistance with x,

$$\rho(x) = \rho^0\exp(\gamma x), \tag{40}$$

the integration of Eq. (39) between points $(x^0, y^0) = (0,0)$ and (x, y) yields an optimal path

$$y(x) = y^0 + \frac{c}{\rho^0}\int_{x^0}^{x}\frac{\mathrm{d}x'}{\exp(\gamma x')} = \frac{c}{\gamma\rho^0}(1-\exp(-\gamma x)). \tag{41}$$

A corresponding slope of Darcy's ray is described by an equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{c}{\rho^0}\exp(-\gamma x). \tag{42}$$

In Eqs. (41) and (42) the ratio c/ρ^0 is the initial slope $(\mathrm{d}y/\mathrm{d}x)^0$ at the point (x^0, y^0). Eq. (42) describes the slope of Darcy's ray decreasing exponentially with x, thus turning toward the direction of the resistance gradient. Indeed, in order to minimize the total resistance, the ray spanned between two given points must take the shape that assures that its relatively large part resides in the 'rarer' region of the medium. This is in agreement with the tangent law of refraction from a rarer to a denser medium in Fig. 2. Eq. (41) proves that as x tends to infinity, y approaches the asymptotic value $c/(\gamma\rho^0)$. For an infinite ratio ρ^0/c, or for the vanishing initial slope $(\mathrm{d}y/\mathrm{d}x)^0$, one obtains $y = 0$ and $\mathrm{d}y/\mathrm{d}x = 0$ for all x. In this case a Darcy's ray initially in the direction of the hydraulic resistance gradient propagates undeviated. Otherwise, considering the inverted form of Eq. (41)

$$x = -\gamma^{-1}\ln(1 - c^{-1}\gamma\rho^0 y) \tag{43}$$

and Eq. (42) one concludes that if $\rho^0 c^{-1} = 0$ then x and $\mathrm{d}x/\mathrm{d}y$ are zero for all y. This means that a ray perpendicular to the resistance gradient (tangent to a surface of the constant resistivity) also propagates undeviated. This is consistent with the discrete tangent law of refraction, but, as first pointed out by Tan and Holland [1], this is alien to Snell's law because in geometrical optics a ray always bends toward the gradient of the index of refraction [8].

In an opposite case the thermal resistivity can be an exponentially decreasing function of x

$$\rho(x) = \rho^0\exp(-\gamma x), \tag{44}$$

the corresponding formulae follow from the previous ones when γ is replaced by $-\gamma$. Eqs. (41) and (42) take respectively the form

$$y(x) = y^0 + \frac{c}{\rho^0}\int_{x^0}^{x}\frac{\mathrm{d}x'}{\exp(-\gamma x')} = \frac{c}{\gamma\rho^0}(\exp(\gamma x) - 1). \tag{45}$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{c}{\rho^0}\exp(\gamma x). \tag{46}$$

The slope of Darcy's ray increases exponentially with x, bending away from the direction of the initial slope $(dy/dx)^0 = c/\rho^0$. This is in agreement with the tangent law of refraction from a denser to a rarer medium. In this case there is no asymptotic value of y. For a vanishing initial slope c/ρ^0 one obtains $y = 0$ and $dy/dx = 0$ for all x which means that a Darcy's ray initially in the direction of the resistance gradient propagates undeviated. Otherwise, considering the inverted forms of Eqs. (45) and (46) one concludes that x and dx/dy are zero for all y if the initial slope is infinite ($\rho^0/c = 0$). This means that a ray perpendicular to the resistance gradient (tangent to a surface of the constant resistance) also propagates undeviated. Again, while this is consistent with the tangent law, it is alien to Snell's law. (In geometrical optics a ray always bends towards the gradient of refraction index.)

7. Concluding remarks

In this paper we have shown applications of the macroscopic variational theory to various practical systems with mass transfer, e.g. to porous drying systems, in which a non-uniform distribution of fluid's mass flux is observed, caused by local variations of porosity and hydraulic resistance (imperfect systems). The results show that a sort of refraction law governs the fluid conduction through an inhomogeneous porous skeleton, the so-called tangent law of bending. It was first found in studies of boundary conditions for the heat flow through a discontinuity at which the thermal conductivity has a jump [1,2]. The tangent law is different from Snell's law of refraction in optics, with the tangents of the angles of incidence and refraction replacing the sines and the conductivity reciprocal taking the place of the refractive index. Such law is known for the electric field intensity at the boundary between two dielectrics [8], and it also applies to potential fields in general [1]. By considering in Section 6 the example when the flow resistance increases exponentially with x (the porous system becoming "denser" with x) one shows that the slope of the steady Darcy's ray decreases with x, thus turning toward the direction of the specific resistance gradient. Indeed, by minimizing the total resistance, the ray spanned between two given points takes the shape that assures that its relatively large part resides in a 'rarer' region of the porous medium, i.e. a region of its low resistance. In other words, in inhomogeneous porous system, the fluid path bends into the direction that ensures its shape corresponding with the longest residence time of the fluid in regions of lower resistance. This makes one possible to predict shapes of Darcy's rays or paths of the fluid flow. This also leads to description of the fluid flow in terms of wave-fronts and corresponding Hamilton–Jacobi theory [10]. It can also be shown that a Darcy's ray initially in the direction of the resistivity gradient propagates undeviated. However, since the tangent law holds, a ray perpendicular to the resistivity gradient (tangent to a layer of the constant resistivity) also propagates undeviated. This is consistent with the tangent law of bending but this is not in agreement with Snell's law where a ray always bends toward the gradient of the index of refraction. These results represent basic characteristics of steady nonlinear transfer of fluids in frictional porous media.

The so-called constructal theory of Bejan [11,12], allows the imbedding of these results into a broader framework of systems with optimal geometry assuring quickest access between a finite-size volume and one point. Frictional generalizations of Euler equation of fluid motion [13] imply that Darcy's law is valid not only to unidirectional flows (for which it was experimentally established) but also to three-dimensional flows; consistent with the results obtained here. Yet there are limitations of the method presented. First, any extension of the variational methodology to unsteady flows is extremely difficult and requires the use of entirely different methods [14]. Second, nonlinearities in derivatives of state variables and abrupt changes of state are excluded. Third, only very slow, linearly damped laminar flows (creeping flows) are correctly described (Darcy Reynolds number $<$ 10). In fact, Darcy's law has been found to be invalid for high values of Reynolds number (turbulent flows). Darcy's law has been found to be invalid either at very low values of hydraulic gradient in some very low-permeability materials, such as clays [15,16] and in some other media, such as ceramic foams [17]. A review of these issues is available in Ref. [18].

On the other hand, some generalized equations for transport in porous media, such as the Dupuit-Forchheimer equation, Darcy-Weisbach equation or Ergun's equation [19–23], can describe both laminar or turbulent flows through the bed. Extension of our approach to these generalized physical situations is a challenging issue, even for steady systems. Especially systems in which fluid flow is nonlinear with respect to the pressure gradient would constitute a valuable and interesting extension.

The present results imply applications to various practical systems with mass transfer where non-uniform distribution of fluid's mass flux is caused by local variations of porosity and permeability ("imperfect systems"). Examples are various groundwater flows, drinking wells, porous drying systems, petroleum ground systems, etc., where most problems involve flow and solute transport. A special situation arises in coastal aquifers, where excess pumping may cause seawater intrusion, thus threatening the use of the aquifer as a source of fresh water. Modeling flow and pollution of groundwater and seawater intrusion in coastal aquifers are recent, attractive applications [24,25].

Let us now explain what effect our results could have on the practical formulation of the above problems concerning flows of fluids in inhomogeneous media satisfying Darcy's law. Effective management of water resources requires the ability to forecast the response of the managed system, e.g., an aquifer, to a collection of operations, such as pumping, recharging, and control of operational variables at aquifer boundaries. Any planning of mitigation, cleanup, or control, requires forecasting the trajectory and history of the fluid and contaminants in both the unsaturated zone and the aquifer. The prediction tool is the numerical model that simulates the flow and pollution motion and transformation. The construction of suitable models should be based on good understanding of the phenomena within the modeled domain and on the domain's boundaries, including chemical and biological processes, on the ability to express this information in the compact form of a well posed formal, possibly analytical model, and, eventually, in the form of its numerical counterpart contained in the related computer program. By applying suitable numerical techniques, a computer leads to the solution describing predictions of the response of the investigated domain to planned activities. For flow and solute transport in the subsurface many suitable programs are now available. Whenever a porous system is inhomogeneous and its resistance ρ (or associated permeability) is a given function of the space variables x and y the computer program should contain the recurrence equation of dynamic programming, Eq. (25) of the present paper, which solves the problem of the fluid's flow trajectory governed by nonlinear Darcy's law and replaces Darcy's equation itself in the computational program. Let us also note that in problems with undetermined resistance functions that are set spontaneously in the process (such as, for example, for Darcy's flows in fluidized beds) Eq. (25) should be solved simultaneously by a trial and error procedure along with balance laws for a fluid in a porous medium. This is, in fact, a general feature in many complex problems where the determination of Darcy's trajectories constitutes merely a component in the complete solution describing a practical process in a inhomogeneous porous system, the other components being: the balance and kinetic equations for energy exchange, equations of balance and movement for the contaminants and equations of pollution transformation by chemical reactions.

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